Geometric Origin of CP Violation in an Extra-Dimensional Brane World

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Abstract

The fermion mass hierarchy and finding a predictive mechanism of the flavor mixing parameters remain two of the least understood puzzles facing particle physics today. In this work, we demonstrate how the realization of the Dirac algebra in the presence of two extra spatial dimensions leads to complex fermion field profiles in the extra dimensions. Dimensionally reducing to four dimensions leads to complex quark mass matrices in such a fashion that CP violation necessarily follows. We also present the generalization of the Randall-Sundrum scenario to the case of a multi-brane, six-dimensional brane-world and discuss how multi-brane worlds may shed light on the generation index of the SM matter content.

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1 Introduction

A fundamental explanation of both the quark flavor-mixing matrix and the fermion masses and their hierarchical structure persist to be two of the most challenging problems of particle physics today. A predictive mechanism for

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fermion mass generation is currently lacking. After spontaneous symmetry breaking, the quark mass term in the langrangian reads:

$$\mathcal{L}_{mass}=\frac{v}{\sqrt{2}}\left(\overline{u_{L_i}}h_{ij}^{(u)}u_{R_j}+\overline{d_{L_i}}h_{ij}^{(d)}d_{R_j}\right)+\text{h.c.} \tag{1}$$

where the h_{ij} are arbitrary 3×3 complex Yukawa coupling matrices. Within the standard model (SM) the fermion masses, the quark flavor-mixing angles and the CP violating phase are free parameters and no relation exists among them. The SM can accommodate the observed mass spectrum of the fermions but unfortunately does not predict it. Thus the calculability of the fermion masses remains an outstanding theoretical challenge. Our hope is that some predictive mechanism of fermion mass generation exists and will place the understanding of fermion mass on a par with that of gauge boson mass.

The number of free parameters in the Yukawa sector eliminates any real predictive power of this sector of the SM. A first step one may take is to modify the Yukawa sector in such a way that the four quark flavor-mixing parameters depend solely on the quark masses themselves. As an attempt to derive a relationship between the quark masses and flavor-mixing parameters, mass matrix ansätze based on flavor democracy with a suitable breaking so as to allow mixing between the quarks of nearest kinship via nearest neighbor interactions was suggested about two decades ago [?, ?, ?, ?, ?, ?, ?, ?]. These early attempts are the first examples of "strict calculability"; i.e., mass matrices such that all flavor-mixing parameters depend solely on, and are determined by, the quark masses. But the simple symmetric NNI texture leads to the experimentally violated inequality $M_{top} < 110$ GeV, prompting consideration of a less restricted form for the mass matrices so as to retain calculability, yet be consistent with experiment [?].

After implementing this first step towards gaining a deeper understanding of the Yukawa sector of the SM in the guise of calculability, one may then attempt to confront the fundamental problem of explaining the fermion mass hierarchy itself. In this paper, we will address neither of the above issues, but instead shall retreat even further into a simpler domain of the overall problem. The quark mixing matrix contains four physical parameters, the three mixing angles and the single CP violating phase of the Cabibo-Kobayashi-Maskawa (CKM) matrix. Here we conjecture that the CP phase in the quark flavor-mixing matrix may be a result of the existence of extra dimensions and the Dirac algebra realized in the presence of these extra dimensions. The notion of CP violation arising from the presence of extra dimensions is not new, but was studied long ago [?], and more recently in [?, ?, ?, ?, ?, ?, ?].

Our work is largely inspired by the papers [?, ?, ?, ?] in which the fermion mass and mixing hierarchies have been addressed within the context of large extra dimensions (LED), both in the five- and six-dimensional cases, as well as within the context of a single extra dimension with warped geometry. This paper addresses the same problem using two extra dimensions with warped geometry, so as to explain the existence of the CP phase in the quark-flavor mixing matrix. Six dimensional extensions of the RS scenario have been studied in [?, ?, ?, ?, ?].

Another major theoretical problem in physics is that of the hierarchy between the Planck scale (the rest mass of a flea) and particle physics scales, such as the masses of the W and Z bosons. String/M-theory seems to be the most promising candidate to form a tight conceptual connection between gravity and particle physics, with its attendant extra dimensions. Some novel ideas concerning solutions to the gauge hierarchy problem are rooted in the possibility that the hierarchy is controlled by exotic features of the extra dimensions; namely, either they are very large and so the hierarchy is generated by the volume of the extra dimensions [?, ?, ?] or that the extra dimensions are warped, with the hierarchy generated by an exponential damping [?, ?, ?].

Here we explore the implications of extra dimensions for the fermion mass and mixing problem. This investigation sheds light on the source of CP violation.

The organization of this paper is as follows. We briefly review the Randall-Sundrum scenario and introduce its extension to two extra dimensions. We then introduce fundamental six-dimensional fermions (eight-component objects), as well as a fundamental Higgs scalar. We present the equations for the fermion zero modes, as well as the boundary conditions. Previous treatments of the fermion boundary conditions in six dimensions differ from the ones presented here. We solve for the Higgs zero mode and show, for particular values of its six-dimensional mass, how its profile is peaked away from the six dimensional analogue of the hidden brane in the brane set-up to be described. The possibility of bulk SM fields within the RS scenario has been extensively studied in [?]. We demonstrate that the presence of two extra dimensions leads to complex fermion field profiles in the extra dimension and show how this leads to CP violation. We then conclude and briefly discuss some future avenues to be investigated.

2 Many-Brane, Six-Dimensional Extension of the Randall-Sundrum Solution

Here we present a simple generalization of the RS scenario to the six dimensional, multi-brane case. In what follows, we consider one fundamental cell of the brane lattice (to be described below), but for completeness we write down the entire brane-lattice solution. Ultimately, one wants to understand not only the fermion mass hierarchy and the flavor mixing parameters, but also the very existence of the three fermion generations of the SM. An interesting observation of Kogan et al [?] is that in multi-brane worlds, there exist ultra-light localized and strongly coupled bulk fermion KK modes. This leads to the possibility that for a given fundamental bulk fermion field with given SM gauge group transformation properties, the generation index may be associated with KK mode number, so that ultimately there is only one six-dimensional species of up-type quarks, for example, and that the generation structure is just a reflection of the existence of ultra-light KK modes arising from the brane set-up and geometry of the extra dimensions. One stumbling block confronting this mode of understanding fermion generation structure is that the very multi-brane set-up that gives rise to the family structure also gives rise to the same number of ultralight KK modes for all other fields, including the graviton. This approach to the family index is currently under investigation, though in the applications to follow, we will consider only one fundamental cell of this brane crystal and hence will not have any ultralight KK modes.

We now present a solution for the metric corresponding to a six dimensional, multi-brane extension of the RS scenario. A five dimensional multi-brane extension is presented in [**?**].

To generalize to six dimensions, we consider a N $\times$ M lattice of N parallel 4-branes localized in the ϕ dimension orthogonal to M parallel 4-branes localized in the ρ dimension. 3-Branes reside at their loci of intersection. The action describing this set-up is given by the following three terms:

$$S = S_{gravity} + \sum_{i=1}^{N} S_i + \sum_{j=1}^{M} S_j$$

where S_{grav} is

$$S_{grav} = \int d^4x \int_0^{2\pi} d\phi \int_0^{2\pi} d\rho \sqrt{-g} \left(\frac{1}{\kappa_6^2} R - \sum_{i,j} \Lambda_{ij} \left[\Theta\left(\phi - \phi_i\right) - \Theta\left(\phi - \phi_{i+1}\right) \right] \times \right.$$
$$\left. \left[\Theta\left(\rho - \rho_j\right) - \Theta\left(\rho - \rho_{j+1}\right) \right] \right)$$

and the terms in the action representing the 4-branes are

$$S_i = - \int d^4x \int_0^{2\pi} d\rho \sqrt{g^{(\phi=\phi_i)}} T_{\phi_i}$$

and

$$S_j = - \int d^4x \int_0^{2\pi} d\phi \sqrt{g^{(\rho=\rho_j)}} T_{\rho_j},$$

where the T_{ϕ_i} are the tensions of the 4-branes located at ϕ_i and T_{ρ_j} are the tensions of the 4-branes located at ρ_j. We are intersted in the case of both extra dimensions being compact and impose the following S^1 periodicity conditions on the extra coordinates:

$$\rho_{M+1} = 2\pi$$
$$\phi_{N+1} = 2\pi$$

Furthermore, we consider an orbifold by imposing a pair of Z_2 symmetries on the solution. As in [**?**], we use the S^1 symmetry(ies) to define the position of the first set of brane sources to be at the origin of the extra dimensions,

$$\phi_1 = 0 = \rho_1$$

In the above expressions for the 4-brane actions, the induced metric is

$$g_{ab}^{\rho=\rho_j} = g_{ab}\left(x^\mu, \phi, \rho = \rho_j\right)$$

for the 4-branes localized in the ϕ direction and by

$$g_{\alpha\beta}^{\phi=\phi_i} = g_{\alpha\beta}\left(x^\mu, \phi = \phi_i, \rho\right)$$

for the 4-branes localized in the ρ direction. The six dimensional Einstein equations are given by

$$R_N^M - \frac{1}{2}\delta_N^M R = \frac{\kappa_6^2}{2} T_N^M$$

where

$$\begin{aligned}
T_N^M &= -\sum_{i,j}^{N,M} \Lambda_{ij}\left[\Theta\left(\phi-\phi_i\right) - \Theta\left(\phi - \phi_{i+1}\right)\right]\left[\Theta\left(\rho-\rho_j\right) - \Theta\left(\rho-\rho_{j+1}\right)\right]\delta_N^M \\
&= -\sum_i^N \sqrt{\frac{-\det g^{\phi=\phi_i}}{\det g}} T_{\phi_i}\delta\left(\phi-\phi_i\right)\delta_a^M\delta_N^a \\
&= -\sum_{j=1}^M \sqrt{\frac{-\det g^{\rho=\rho_j}}{\det g}} T_{\rho_j}\delta\left(\rho-\rho_j\right)\delta_\alpha^M\delta_N^\alpha
\end{aligned}$$

We are not addressing any cosmological issues in this work, and for simplicity consider the following static ansätze for the metric:

$$ds^2 = A^2\left(\phi,\rho\right)\eta_{\mu\nu}dx^\mu dx^\nu - B^2\left(\phi,\rho\right)d\phi^2 - C^2\left(\phi,\rho\right)d\rho^2$$

With this ansätz, the left-hand side of the Einstein equations are given by the following expressions, where dots denote derivatives with respect to the ρ coordinate and primes denote derivatives with respect to the ϕ coordinate:

$$\begin{aligned}
G_\nu^\mu &= \frac{2}{A^2}\left[\left(\frac{\dot{A}}{A}\right)^2 + \left(\frac{A'}{A}\right)^2 + 2\left(\frac{\ddot{A}}{A} + \frac{A''}{A}\right)\right]\delta_\nu^\mu \\
G_\phi^\phi &= \frac{2}{A^2}\left[5\left(\frac{A'}{A}\right)^2 + 2\frac{\ddot{A}}{A} + \left(\frac{\dot{A}}{A}\right)^2\right]
\end{aligned}$$

$$\begin{aligned}
G_\rho^\rho &= \frac{2}{A^2}\left[5\left(\frac{\dot{A}}{A}\right)^2 + 2\frac{A''}{A} + \left(\frac{A'}{A}\right)^2\right] \\
G_\rho^\phi &= \frac{4}{A^4}\left[-A\dot{A}' + 2\dot{A}A'\right] \\
G_\phi^\rho &= G_\rho^\phi
\end{aligned}$$

Taking the warp factor to be

$$A = \frac{1}{e^{\sigma(\phi)} + e^{\gamma(\rho)} + 1}$$

and

$$B(\phi,\rho) = A(\phi,\rho)\, e^{\sigma(\phi)}$$
$$C(\phi,\rho) = A(\phi,\rho)\, e^{\gamma(\rho)}$$

one can easily check that the nondiagonal elements of the Einstein tensor vanish, $G^{\tilde{\phi}}_{\tilde{\rho}} = 0$, and thus the $\tilde{\phi} - \tilde{\rho}$ component of the Einstein equations are trivially satisfied.

The remaining equations will be satisfied if the following relations are fulfilled:

$$10\left(k^2_{\phi_i} + k^2_{\rho_j}\right) = -\frac{\kappa_6^2}{2}\sum_{i,j}\Lambda_{ij}\left[\Theta(\phi-\phi_i) - \Theta(\phi-\phi_{i+1})\right]\times$$
$$\left[\Theta(\rho-\rho_j) - \Theta(\rho-\rho_j)\right],$$
$$8\left(k_{\rho_j} - k_{\rho_{j-1}}\right) = \frac{\kappa_6^2}{2}T_{\rho_j},$$
$$8\left(k_{\phi_i} - k_{\phi_{i-1}}\right) = \frac{\kappa_6^2}{2}T_{\phi_i},$$

So we have

$$\sigma(\phi) = k_{\phi_1}|\phi-\phi_1|\Theta(\phi-\phi_1) + \left(k_{\phi_2} - k_{\phi_1}\right)|\phi-\phi_2|\Theta(\phi-\phi_2) +$$
$$\left(k_{\phi_3} - k_{\phi_2}\right)|\phi-\phi_3|\Theta(\phi-\phi_3) + \cdots + \left(k_{\phi_N} - k_{\phi_{N-1}}\right)|\phi-\phi_N|\Theta(\phi-\phi_N)$$
$$\gamma(\rho) = k_{\rho_1}|\rho-\rho_1|\Theta(\rho-\rho_1) + \left(k_{\rho_2} - k_{\rho_1}\right)|\rho-\rho_2|\Theta(\rho-\rho_2) +$$
$$\left(k_{\rho_3} - k_{\rho_2}\right)|\rho-\rho_3|\Theta(\rho-\rho_3) + \cdots + \left(k_{\rho_M} - k_{\rho_{M-1}}\right)|\rho-\rho_M|\Theta(\rho-\rho_M)$$

and

$$k_i - k_{i-1} = \frac{\kappa_6^2}{16}T_i$$
$$\kappa_6^2 = \frac{16\pi}{M_6^4}$$

We work in units where $M_6^{-4} = 1$

$$k_i - k_{i-1} = \pi T_i$$

and for simplicity, we take all the bulk cosmological constants to be equal and the magnitudes of the brane tensions to be the same: $|T_{\phi_i}| = |T_{\rho_j}| = T$ Thus we have the relations

$$10\,(2)\,k_1^2 = -\frac{16\pi}{(2)\,M_6^4}\Lambda$$

$$k_1 = \sqrt{-\frac{2\pi\Lambda}{5}} = k_{\phi_1} = k_{\rho_1}$$

Our expressions for the functions $\sigma(\phi)$ and $\gamma(\rho)$ then become

$$\sigma(\phi) = \sqrt{-\frac{2\pi\Lambda}{5}}\phi\Theta(\phi) + \pi T\left[-(\phi-\phi_2)\Theta(\phi-\phi_2)\right.$$
$$+(\phi-\phi_3)\Theta(\phi-\phi_3) - (\phi-\phi_4)\Theta(\phi-\phi_4)$$
$$+(\phi-\phi_5)\Theta(\phi-\phi_5) - (\phi-\phi_6)\Theta(\phi-\phi_6)$$
$$\left.+\ldots-(\phi-\phi_N)\Theta(\phi-\phi_N)\right]$$
$$\sigma(\phi_c) = \sigma(0) = 0 \rightarrow$$
$$\phi_c = \frac{\pi T(\phi_2-\phi_3+\phi_4-\phi_5+\phi_6-\ldots+\phi_N)}{\left(\pi T - \sqrt{-\frac{2\pi\Lambda}{5}}\right)}$$

and similarly,

$$\gamma(\rho) = \sqrt{-\frac{2\pi\Lambda}{5}}\rho\Theta(\rho) + \pi T\left[-(\rho-\rho_2)\Theta(\rho-\rho_2)\right.$$
$$\left.+\ldots-(\rho-\rho_M)\Theta(\rho-\rho_M)\right]$$
$$\gamma(\rho_c) = \gamma(0) = 0 \rightarrow$$
$$\rho_c = \frac{\pi T(\rho_2-\ldots+\rho_M)}{\left(\pi T - \sqrt{-\frac{2\pi\Lambda}{5}}\right)}.$$

3 Six Dimensional Dirac Fermions

In the seminal work of [?], a framework is introduced for understanding both the fermion mass hierarchy and proton stability without recourse to flavor symmetries in terms of higher dimensional geography. Additional physics must be assumed in order to localize the fermions in the flat extra dimension, but once this additional scalar field is introduced, any coupling between chiral fermions is exponentially suppressed because the two fields are separated in space. A key observation made by Huber and Shafi in [?] is that one can get this exponential damping automatically from a non-factorizable geometry and that there is no need to assume additional physics.

In both [?] and [?], the effective four dimensional masses arise from integrating over the extra dimension Yukawa interactions between the five dimensional Higgs and chiral fermion fields. The resulting four dimensional Yukawa coupling matrices exhibit phenomenologically acceptable spectrums and mixing (excluding CP violation) because of how these overlap integrals can be tuned depending on the values of the five dimensional mass term for the fermions in the action. In the flat space scenario of [?], the five dimensional mass term serves to translate the gaussian profile of the fermions along the extra dimension, so that the overlap integral of two chiral fermions and the flat zero mode of the Higgs is itself a gaussian, thus generating exponentially small effective four dimensional Yukawa couplings. In the warped space scenario of [?], with the $\frac{S^1}{Z_2}$ geometry of Randall-Sundrum, the right-handed zero modes are peaked at the origin on the positive tension 3-brane, while the left-handed zero modes are peaked at the other orbifold fixed point around the negative tension 3-brane. In this case, varying the value of the five dimensional mass does not translate the fermion field profiles along the extra dimension; the right-handed and left-handed fermions remain localized around the positive and negative tension 3-branes, respectively. What does change, however, is the width of the profile. Hence, just as in the flat space scenario, small changes in the values of the five dimensional mass parameters lead to greatly amplified changes in the resulting four dimensional Yukawa coupling matrices.

In both the flat and warped space scenarios in five dimensions, the profiles for all fields in the extra dimension are real and CP violation cannot be realized in a natural way. While CP violation is not addressed in [?], a numerical example is given where nine input parameters, essentially the ratios of the five dimensional masses for the fermions appearing in the five dimensional action to the AdS curvature scale, result in very good agreement with the quark mass spectrum and absolute values of the elements of the V_{CKM} matrix. The agreement is striking, with just enough disagreement in the mixing matrix to generate the suspicion that inclusion of CP violation may somehow bring about even closer agreement between this type of model's predictions and experiment.

The action for a fermion in the six dimensional background considered here is [?]

$$S = \int d^4x \int d\phi d\rho \sqrt{G} \left\{ E^A_\alpha \left[\frac{i}{2} \overline{\psi} \gamma^\alpha \left(\overrightarrow{\partial}_A - \overleftarrow{\partial}_A \right) \psi \right] - m\left(\phi, \rho\right) \overline{\psi}\psi \right\} \tag{2}$$

$$ds^2 = A^2\left(\phi, \rho\right) \eta_{\mu\nu} dx^\mu dx^\nu - B^2\left(\phi, \rho\right) d\phi^2 - C^2\left(\phi, \rho\right)^2 \tag{3}$$

$$A = \frac{1}{e^\sigma + e^\gamma - 1} \tag{4}$$

$$B = e^\sigma A \tag{5}$$

$$C = e^\gamma A \tag{6}$$

$$\sqrt{g} = \frac{e^\sigma e^\gamma}{\left(e^\sigma + e^\gamma - 1\right)^6} \tag{7}$$

We now show that the introduction of another extra dimension leads to a natural explanation of CP violation. In six spacetime dimensions the Dirac algebra is minimally realized by 8×8 matrices. A particularly convenient representation is that of [?], in which the ideas of [?] have been extended to the six dimensional case. This provides a theoretical motivation for the so-called democratic mass matrices that have served as the starting point for many flavor symmetry approaches to the quark mass hierarchy problem. This convenient representation is presented below, where the γ_5 in Γ_ϕ is the same γ_5 constructed from the four dimensional γ's.

$$\Gamma_0 = \begin{pmatrix} 0 & +i\gamma_0 \\ -i\gamma^0 & 0 \end{pmatrix} \tag{8}$$

$$\Gamma_1 = \begin{pmatrix} 0 & +i\gamma_1 \\ -i\gamma^1 & 0 \end{pmatrix} \tag{9}$$

$$\Gamma_2 = \begin{pmatrix} 0 & +i\gamma_2 \\ -i\gamma^2 & 0 \end{pmatrix} \tag{10}$$

$$\Gamma_3 = \begin{pmatrix} 0 & +i\gamma_3 \\ -i\gamma^3 & 0 \end{pmatrix} \tag{11}$$

$$\Gamma_\phi = \begin{pmatrix} 0 & \gamma_5 \\ -\gamma^5 & 0 \end{pmatrix} \tag{12}$$

$$\Gamma_\rho = \begin{pmatrix} 0 & +i_{4\times 4} \\ +i_{4\times 4} & 0 \end{pmatrix} \tag{13}$$

$$\Gamma_7 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{14}$$

$$\gamma_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{15}$$

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{16}$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \tag{17}$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{18}$$

$$\gamma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{19}$$

Both the Γ 's and the γ's have the mostly minus signature, accounting for the discrepancy between this representation and the one presented in [**?**].

Using Γ_7, we can construct projection operators in the usual way to get two four-component objects we call ψ_+ and ψ_-. This procedure is in perfect analogy with what is done in four dimensions (expressing a four dimensional fermion field in terms of its left and right-handed components).

$$\psi_+ = \frac{1}{2}(1-\Gamma_7)\psi \tag{20}$$

$$\psi_- = \frac{1}{2}(1+\Gamma_7)\psi \tag{21}$$

$$\psi = \psi_+ + \psi_- \tag{22}$$

Note that in this representation the six dimensional lorentz invariant fermion bilinear $\overline{\psi}\psi$ has a complex coefficient when expressed in terms of the four component fields ψ_+ and ψ_- and their four component conjugate fields $\overline{\psi_+}$ and $\overline{\psi_-}$. This leads to a real six dimensional mass term which is complex when expressed in terms of the four component projections ψ_+ and ψ_- (the six dimensional analogs of ψ_L and ψ_R).

$$\overline{\psi} = \psi^\dagger\Gamma_0 = \left(\psi_+^\dagger, \psi_-^\dagger\right)\left(\begin{array}{cc} 0 & i\gamma_0 \\ -i\gamma_0 & 0 \end{array}\right) = \left(-i\overline{\psi_-}, i\overline{\psi_+}\right) \tag{23}$$

As in [**?**], we assume that the higher dimensional fermion mass term is the result of the coupling of the fermions with a scalar field which has a nontrivial stable vacuum. We assume this VEV to have a multi-kink solution, which we can express in terms of the functions $\sigma(\phi)$ and $\gamma(\rho)$.

After integrating by parts, we can recast the action as:

$$\begin{aligned}
S \;=\; & \int \mathrm{d}^4x \int \mathrm{d}\phi\mathrm{d}\rho \Bigg(\frac{ie^\sigma e^\gamma}{(e^\sigma+e^\gamma-1)^5}\left[\overline{\psi_{R+}}\gamma^\mu\partial_\mu\psi_{R+} + \overline{\psi_{L+}}\gamma^\mu\partial_\mu\psi_{L+} + \overline{\psi_{R-}}\gamma^\mu\partial_\mu\psi_{R-} + \overline{\psi_{L-}}\gamma^\mu\partial_\mu\psi_{L-}\right] \\
& -\frac{1}{2}\overline{\psi_{R-}}\left(\frac{e^\gamma}{(e^\sigma+e^\gamma-1)^5}\partial_\phi + \partial_\phi\frac{e^\gamma}{(e^\sigma+e^\gamma-1)^5}\right)\psi_{L-} \\
& +\frac{1}{2}\overline{\psi_{L-}}\left(\frac{e^\gamma}{(e^\sigma+e^\gamma-1)^5}\partial_\phi + \partial_\phi\frac{e^\gamma}{(e^\sigma+e^\gamma-1)^5}\right)\psi_{R-} \\
& -\frac{1}{2}\overline{\psi_{R+}}\left(\frac{e^\gamma}{(e^\sigma+e^\gamma-1)^5}\partial_\phi + \partial_\phi\frac{e^\gamma}{(e^\sigma+e^\gamma-1)^5}\right)\psi_{L+} \\
& +\frac{1}{2}\overline{\psi_{L+}}\left(\frac{e^\gamma}{(e^\sigma+e^\gamma-1)^5}\partial_\phi + \partial_\phi\frac{e^\gamma}{(e^\sigma+e^\gamma-1)^5}\right)\psi_{R+} \\
& +\frac{i}{2}\overline{\psi_{R-}}\left(\frac{e^\gamma}{(e^\sigma+e^\gamma-1)^5}\partial_\rho + \partial_\rho\frac{e^\gamma}{(e^\sigma+e^\gamma-1)^5}\right)\psi_{L-} \\
& +\frac{i}{2}\overline{\psi_{L-}}\left(\frac{e^\gamma}{(e^\sigma+e^\gamma-1)^5}\partial_\rho + \partial_\rho\frac{e^\gamma}{(e^\sigma+e^\gamma-1)^5}\right)\psi_{R-} \\
& -\frac{i}{2}\overline{\psi_{R+}}\left(\frac{e^\gamma}{(e^\sigma+e^\gamma-1)^5}\partial_\rho + \partial_\rho\frac{e^\gamma}{(e^\sigma+e^\gamma-1)^5}\right)\psi_{L+}
\end{aligned}$$

$$
-\frac{i}{2}\overline{\psi_{L+}}\left(\frac{e^{\gamma}}{\left(e^{\sigma}+e^{\gamma}-1\right)^{5}}\partial_{\rho}+\partial_{\rho}\frac{e^{\gamma}}{\left(e^{\sigma}+e^{\gamma}-1\right)^{5}}\right)\psi_{R+}
$$

$$
\left.-mi\left(\frac{\sigma'}{k}\right)\left(\frac{\dot{\gamma}}{k}\right)\frac{ie^{\sigma}e^{\gamma}}{\left(e^{\sigma}+e^{\gamma}-1\right)^{5}}\left(\overline{\psi_{L+}}\psi_{R-}+\overline{\psi_{R+}}\psi_{L-}-\overline{\psi_{L-}}\psi_{R+}-\overline{\psi_{R-}}\psi_{L+}\right)\right)
$$

In order to extract the four dimensional physics, we write the action in terms of a sum over KK modes. Ultimately, we will only be interested in the zero modes.

$$
\begin{aligned}
S = & \sum_{m}\sum_{n}\int \mathrm{d}^4x\left\{i\overline{\psi_{n,m+}}\left(x\right)\gamma^{\mu}\partial_{\mu}\psi_{n,m+}\left(x\right)+i\overline{\psi_{n,m-}}\left(x\right)\gamma^{\mu}\partial_{\mu}\psi_{n,m-}\left(x\right)\right. \\
& \left.-m_{n,m+}\overline{\psi_{n,m+}}\left(x\right)\psi_{n,m+}\left(x\right)-m_{n,m-}\overline{\psi_{n,m-}}\left(x\right)\psi_{n,m-}\left(x\right)\right\}
\end{aligned}
$$

The decomposition of the KK modes is simplified when we express ψ_{R+} and ψ_{L+} in the form:

$$
\Psi_{(R,L)+}\left(x,\phi,\rho\right)=\sum_{m}\sum_{n}\psi^{R,L}_{n,m+}\left(x\right)\left(\frac{e^{\sigma}e^{\gamma}}{\left(e^{\sigma}+e^{\gamma}-1\right)^{6}}\right)^{-\frac{1}{2}}f^{R,L}_{n,m+}\left(\phi,\rho\right) \tag{24}
$$

and ψ_{R-} and ψ_{L-} in the form:

$$
\Psi_{(R,L)-}\left(x,\phi,\rho\right)=\sum_{m}\sum_{n}\psi^{R,L}_{n,m-}\left(x\right)\left(\frac{e^{\sigma}e^{\gamma}}{\left(e^{\sigma}+e^{\gamma}-1\right)^{6}}\right)^{-\frac{1}{2}}f^{R,L}_{n,m-}\left(\phi,\rho\right) \tag{25}
$$

To reproduce the standard 4-d kinetic terms, we require the normalization conditions

$$
\int\sum_{k,j}\sum_{n,m}\left(e^{\sigma}+e^{\gamma}-1\right)f^{R^*}_{n,k+}\left(\phi,\rho\right)f^{R}_{m,j+}\left(\phi,\rho\right)\mathrm{d}\phi\mathrm{d}\rho=\delta_{mn},\delta_{kj} \tag{26}
$$

$$
\int\sum_{k,j}\sum_{n,m}\left(e^{\sigma}+e^{\gamma}-1\right)f^{R^*}_{n,k-}\left(\phi,\rho\right)f^{R}_{m,j-}\left(\phi,\rho\right)\mathrm{d}\phi\mathrm{d}\rho=\delta_{mn},\delta_{kj} \tag{27}
$$

$$
\int\sum_{k,j}\sum_{n,m}\left(e^{\sigma}+e^{\gamma}-1\right)f^{L^*}_{n,k+}\left(\phi,\rho\right)f^{L}_{m,j+}\left(\phi,\rho\right)\mathrm{d}\phi\mathrm{d}\rho=\delta_{mn},\delta_{kj} \tag{28}
$$

$$
\int\sum_{k,j}\sum_{n,m}\left(e^{\sigma}+e^{\gamma}-1\right)f^{L^*}_{n,k-}\left(\phi,\rho\right)f^{L}_{m,j-}\left(\phi,\rho\right)\mathrm{d}\phi\mathrm{d}\rho=\delta_{mn},\delta_{kj} \tag{29}
$$

In order to read off the equations of motion that the f's must solve, we need to simplify some terms in the action. For example,

$$
\begin{aligned}
&\frac{1}{2}\overline{\psi_{L+}}\left(\frac{e^{\gamma}}{\left(e^{\sigma}+e^{\gamma}-1\right)^{5}}\partial_{\phi}+\partial_{\phi}\frac{e^{\gamma}}{\left(e^{\sigma}+e^{\gamma}-1\right)^{5}}\right)\psi_{R+} \quad = \\
&-\frac{1}{2}\sum_{l,m=0}^{\infty}\overline{\psi_{L+l,m}}\left(x\right)f^{*}_{L+l,m}\left(\phi,\rho\right)\frac{\left(e^{\sigma}+e^{\rho}-1\right)e^{\gamma}\sigma'}{e^{\sigma}e^{\gamma}}\sum_{n,p=0}^{\infty}\psi_{R+n,p}\left(x\right)f_{R+n,p}\left(\phi,\rho\right) \\
&+3\sum_{l,m=0}^{\infty}\overline{\psi_{L+l,m}}\left(x\right)f^{*}_{L+l,m}\left(\phi,\rho\right)e^{\sigma}e^{\gamma}\sigma'\sum_{n,p=0}^{\infty}\psi_{R+n,p}\left(x\right)f_{R+n,p}\left(\phi,\rho\right) \\
&+\sum_{l,m=0}^{\infty}\overline{\psi_{L+l,m}}\left(x\right)f^{*}_{L+l,m}\left(\phi,\rho\right)\frac{\left(e^{\sigma}+e^{\gamma}-1\right)}{e^{\sigma}}\sum_{n,p=0}^{\infty}\psi_{R+n,p}\left(x\right)f'_{R+n,p}\left(\phi,\rho\right) \\
&-\frac{5}{2}\sum_{l,m=0}^{\infty}\overline{\psi_{L+l,m}}\left(x\right)f^{*}_{L+l,m}\left(\phi,\rho\right)\sigma'\sum_{n,p=0}^{\infty}\psi_{R+n,p}\left(x\right)f_{R+n,p}\left(\phi,\rho\right)
\end{aligned}
$$

with similar expressions holding for the other terms.

We are interested in the equations for the zero modes ($m_n = 0$), found by means of varying S with respect to $f^{*}_{R+0}, f^{*}_{L+0}, f^{*}_{L-0}, f^{*}_{R-0}$.

The normalization conditions reproduce the desired four dimensional kinetic energy terms, hence it is only necessary to vary the remaining parts of S. For example, the remaining terms involving f^{*}_{L+0} are:

$$
\begin{aligned}
&-\frac{1}{2}\overline{\psi_{L+0}}\left(x\right)f^{*}_{L+0}\left(\phi,\rho\right)\frac{\left(e^{\sigma}+e^{\gamma}-1\right)e^{\gamma}}{e^{\sigma}e^{\gamma}}\sigma'\psi_{R+0}\left(x\right)f_{R+0}\left(\phi,\rho\right) \\
&+3\overline{\psi_{L+0}}\left(x\right)f^{*}_{L+0}\left(\phi,\rho\right)e^{\sigma}e^{\gamma}\sigma'\psi_{R+0}\left(x\right)f_{R+0}\left(\phi,\rho\right) \\
&+\overline{\psi_{L+0}}\left(x\right)f^{*}_{L+0}\left(\phi,\rho\right)\frac{\left(e^{\sigma}+e^{\gamma}-1\right)}{e^{\sigma}}\psi_{R+0}\left(x\right)f'_{R+0}\left(\phi,\rho\right) \\
&-\frac{5}{2}\overline{\psi_{L+0}}\left(x\right)f^{*}_{L+0}\left(\phi,\rho\right)\sigma'\psi_{R+0}\left(x\right)f_{R+0}\left(\phi,\rho\right) \\
&+\frac{i}{2}\overline{\psi_{L+0}}\left(x\right)f^{*}_{L+0}\left(\phi,\rho\right)\frac{\left(e^{\sigma}+e^{\gamma}-1\right)2e^{\sigma}\dot{\gamma}}{e^{\sigma}e^{\gamma}}\psi_{R+0}\left(x\right)f_{R+0}\left(\phi,\rho\right) \\
&-3i\overline{\psi_{L+0}}\left(x\right)f^{*}_{L+0}\left(\phi,\rho\right)e^{\gamma}e^{\sigma}\dot{\gamma}\psi_{R+0}\left(x\right)f_{R+0}\left(\phi,\rho\right) \\
&-i\overline{\psi_{L+0}}\left(x\right)f^{*}_{L+0}\left(\phi,\rho\right)\frac{\left(e^{\sigma}+e^{\gamma}-1\right)}{e^{\gamma}}\psi_{R+0}\left(x\right)\dot{f}_{R+0}\left(\phi,\rho\right) \\
&+\frac{i5}{2}\overline{\psi_{L+0}}\left(x\right)f^{*}_{L+0}\left(\phi,\rho\right)\dot{\gamma}\psi_{R+0}\left(x\right)f_{R+0}\left(\phi,\rho\right) \\
&-im\left(\frac{\sigma'}{k}\right)\left(\frac{\dot{\gamma}}{k}\right)\overline{\psi_{L+0}}\left(x\right)f^{*}_{L+0}\left(\phi,\rho\right)\psi_{R-0}\left(x\right)f_{R-0}\left(\phi,\rho\right)
\end{aligned}
$$

Noting that the two four dimensional right-handed and left-handed spinors ψ_+ and ψ_- are complex conjugates of each other [**?**], we arrive at the following equation for the right handed zero mode (recall that dots denote derivatives with respect to ρ and primes denote derivatives with respect to ϕ):

$$
\begin{aligned}
-\frac{1}{2}\frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\sigma}e^{\gamma}}e^{\gamma}\sigma' f_{R+0}+3e^{\sigma}e^{\gamma}\sigma' f_{R+0} & \\
+\frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\sigma}}f'_{R+0}-\frac{5}{2}\sigma' f_{R+0}+\frac{i}{2}\frac{(e^{\sigma}+e^{\gamma}-1)\,e^{\sigma}\dot{\gamma}}{e^{\sigma}e^{\gamma}}f_{R+0} & \\
-3ie^{\gamma}e^{\sigma}\dot{\gamma}f_{R+0}-i\frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\gamma}}\dot{f}_{R+0} & \\
+i\frac{5}{2}\dot{\gamma}f_{R+0}-im\left(\frac{\sigma'}{k}\right)\left(\frac{\dot{\gamma}}{k}\right)f_{R-0} & = 0
\end{aligned}
$$

Similarly, we find the following remaining equations for the other zero modes:

$$
\begin{aligned}
-\frac{1}{2}\frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\sigma}e^{\gamma}}e^{\gamma}\sigma' f_{R-0}+3e^{\sigma}e^{\gamma}\sigma\prime f_{R-0} & \\
+\frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\sigma}}f'_{R-0}-\frac{5}{2}\sigma' f_{R-0}-\frac{i}{2}\frac{(e^{\sigma}+e^{\gamma}-1)\,e^{\sigma}\dot{\gamma}}{e^{\sigma}e^{\gamma}}f_{R-0} & \\
+3ie^{\gamma}e^{\sigma}\dot{\gamma}f_{R-0}+i\frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\gamma}}\dot{f}_{R-0} & \\
-i\frac{5}{2}\dot{\gamma}f_{R-0}+im\left(\frac{\sigma'}{k}\right)\left(\frac{\dot{\gamma}}{k}\right)f_{R+0} & = 0
\end{aligned}
$$

$$
\begin{aligned}
+\frac{1}{2}\frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\sigma}e^{\gamma}}e^{\gamma}\sigma' f_{L+0}-3e^{\sigma}e^{\gamma}\sigma\prime f_{L+0} & \\
-\frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\sigma}}f'_{L+0}+\frac{5}{2}\sigma' f_{L+0}+\frac{i}{2}\frac{(e^{\sigma}+e^{\gamma}-1)\,e^{\sigma}\dot{\gamma}}{e^{\sigma}e^{\gamma}}f_{L+0} & \\
-3ie^{\gamma}e^{\sigma}\dot{\gamma}f_{L+0}-i\frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\gamma}}\dot{f}_{L+0} & \\
+i\frac{5}{2}\dot{\gamma}f_{L+0}-im\left(\frac{\sigma'}{k}\right)\left(\frac{\dot{\gamma}}{k}\right)f_{L-0} & = 0
\end{aligned}
$$

$$
\begin{aligned}
+\frac{1}{2}\frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\sigma}e^{\gamma}}e^{\gamma}\sigma' f_{L-0}-3e^{\sigma}e^{\gamma}\sigma' f_{L-0} & \\
-\frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\sigma}}f'_{L-0}+\frac{5}{2}\sigma' f_{L-0}-\frac{i}{2}\frac{(e^{\sigma}+e^{\gamma}-1)\,e^{\sigma}\dot{\gamma}}{e^{\sigma}e^{\gamma}}f_{L-0} & \\
+3ie^{\gamma}e^{\sigma}\dot{\gamma}f_{L-0}+i\frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\gamma}}\dot{f}_{L-0} & \\
-i\frac{5}{2}\dot{\gamma}f_{L-0}+im\left(\frac{\sigma'}{k}\right)\left(\frac{\dot{\gamma}}{k}\right)f_{L+0} & = 0
\end{aligned}
$$

Because $f^{*}_{L+0}=f_{L-0}$, this is the complex conjugate of the previous equation.

The equations for the right-handed zero modes are

$$\begin{aligned} -\frac{1}{2}\frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\sigma}e^{\gamma}}e^{\gamma}\sigma' f_{R-} + 3e^{\sigma}e^{\gamma}\sigma' f_{R-} + \frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\sigma}}f'_{R-} & \\ -\frac{5}{2}\sigma' f_{R-} - i\frac{1}{2}\frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\sigma}e^{\gamma}}e^{\sigma}\dot{\gamma} f_{R-} + 3ie^{\gamma}e^{\sigma}\dot{\gamma} f_{R-} & \\ +i\frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\gamma}}\dot{f}_{R-} - i\frac{5}{2}\dot{\gamma} f_{R-} + im\left(\frac{\sigma'}{k}\right)\left(\frac{\dot{\gamma}}{k}\right)f_{R+} &= 0 \end{aligned}$$

$$\begin{aligned} -\frac{1}{2}\frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\sigma}e^{\gamma}}e^{\gamma}\sigma' f_{R+} + 3e^{\sigma}e^{\gamma}\sigma' f_{R+} + \frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\sigma}}f'_{R+} & \\ -\frac{5}{2}\sigma' f_{R+} + i\frac{1}{2}\frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\sigma}e^{\gamma}}e^{\sigma}\dot{\gamma} f_{R+} - 3ie^{\gamma}e^{\sigma}\dot{\gamma} f_{R+} & \\ -i\frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\gamma}}\dot{f}_{R+} + i\frac{5}{2}\dot{\gamma} f_{R+} - im\left(\frac{\sigma'}{k}\right)\left(\frac{\dot{\gamma}}{k}\right)f_{R-} &= 0 \end{aligned}$$

Because $f_{R+} = f^*_{R-}$, we are free to write

$$f_{R+} = U + iV \tag{30}$$

$$f_{R-} = U - iV \tag{31}$$

where U and V are real. Setting the real and imaginary parts of this zero mode equation to zero, we find equations:

$$\begin{aligned} -\frac{1}{2}\frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\sigma}e^{\gamma}}e^{\gamma}\sigma' U_R + 3e^{\sigma}e^{\gamma}\sigma' U_R + \frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\sigma}}U'_R & \\ -\frac{5}{2}\sigma' U_R - \frac{1}{2}\frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\sigma}e^{\gamma}}e^{\sigma}\dot{\gamma} V_R + 3e^{\gamma}e^{\sigma}\dot{\gamma} V_R + \frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\gamma}}\dot{V}_R & \\ -\frac{5}{2}\dot{\gamma} V_R - m\left(\frac{\sigma'}{k}\right)\left(\frac{\dot{\gamma}}{k}\right)V_R &= 0 \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2}\frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\sigma}e^{\gamma}}e^{\gamma}\sigma' V_R - 3e^{\sigma}e^{\gamma}\sigma' V_R - \frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\sigma}}V'_R & \\ +\frac{5}{2}\sigma' V_R - \frac{1}{2}\frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\sigma}e^{\gamma}}e^{\sigma}\dot{\gamma} U_R + 3e^{\gamma}e^{\sigma}\dot{\gamma} U_R + \frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\gamma}}\dot{U}_R & \\ -\frac{5}{2}\dot{\gamma} U_R + m\left(\frac{\sigma'}{k}\right)\left(\frac{\dot{\gamma}}{k}\right)U_R &= 0 \end{aligned}$$

Similarly setting the real and imaginary parts of the left-handed zero mode equation to zero, we find:

$$\frac{1}{2}\frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\sigma}e^{\gamma}}e^{\gamma}\sigma' U_L - 3e^{\sigma}e^{\gamma}\sigma' U_L - \frac{(e^{\sigma}+e^{\gamma}-1)}{e^{\sigma}}U'_L$$

$$
\begin{aligned}
+\frac{5}{2}\sigma' U_L - \frac{1}{2}\frac{(e^\sigma + e^\gamma - 1)}{e^\sigma e^\gamma} e^\sigma \dot{\gamma} V_L + 3 e^\gamma e^\sigma \dot{\gamma} V_L + \frac{(e^\sigma + e^\gamma - 1)}{e^\gamma}\dot{V}_L & \\
-\frac{5}{2}\dot{\gamma} V_L - m\left(\frac{\sigma'}{k}\right)\left(\frac{\dot{\gamma}}{k}\right) V_L &= 0
\end{aligned}
$$

and

$$
\begin{aligned}
-\frac{1}{2}\frac{(e^\sigma + e^\gamma - 1)}{e^\sigma e^\gamma} e^\gamma \sigma' V_L + 3 e^\sigma e^\gamma \sigma' V_L + \frac{(e^\sigma + e^\gamma - 1)}{e^\sigma} V_L' & \\
-\frac{5}{2}\sigma' V_L - \frac{1}{2}\frac{(e^\sigma + e^\gamma - 1)}{e^\sigma e^\gamma} e^\sigma \dot{\gamma} U_L + 3 e^\gamma e^\sigma \dot{\gamma} U_L + \frac{(e^\sigma + e^\gamma - 1)}{e^\gamma}\dot{U}_L & \\
-\frac{5}{2}\dot{\gamma} U_L + m\left(\frac{\sigma'}{k}\right)\left(\frac{\dot{\gamma}}{k}\right) U_L &= 0
\end{aligned}
$$

In order to find solutions that respect the symmetries $\phi \to -\phi$ and $\rho \to -\rho$ of this brane background, we need to know how these two Z_2 symmetries are realized on fermions in this background:

$$
\psi\left(x^\mu, \phi, \rho\right) \to \Psi\left(x^\mu, \phi, \rho\right) = \pm \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^\rho \psi\left(x^\mu, \phi_c - \phi, \rho\right) \tag{32}
$$

and

$$
\psi\left(x^\mu, \phi, \rho\right) \to \Psi\left(x^\mu, \phi, \rho\right) = \pm \Gamma^\rho \Gamma^7 \psi\left(x^\mu, \phi, \rho_c - \rho\right) \tag{33}
$$

Because only fermion bilinears appear in the action, the choice of sign is arbitrary. In order to derive definite boundary conditions, we choose the signs given below. One can easily show that

$$
S_\phi = -\Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^\rho \tag{34}
$$

satisfies

$$
S_\phi^{-1} \Gamma^M S_\phi = \Lambda_N^M \Gamma^N \tag{35}
$$

where Λ corresponds to the Lorentz transformation

$$
\Lambda = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \tag{36}
$$

and

$$
L = i\overline{\psi}\Gamma^N \partial_N \psi \tag{37}
$$

is invariant.

Similarly,

$$
S_\rho = -\Gamma_\rho \Gamma_7 \tag{38}
$$

satisfies

$$S_\rho^{-1}\Gamma^M S_\rho = \Lambda_N^M \Gamma^N \tag{39}$$

where Λ corresponds to the L.T.

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{40}$$

and

$$L = i\overline{\psi}\Gamma^N \partial_N \psi \tag{41}$$

is invariant.

With $S_\rho = -\Gamma_\rho \Gamma_7$, this symmetry translates into

$$\begin{pmatrix} \psi'_+ \\ \psi'_- \end{pmatrix} = -\Gamma_\rho \Gamma_7 \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \tag{42}$$

$$\begin{pmatrix} \psi'_+ \\ \psi'_- \end{pmatrix} = \begin{pmatrix} +i\psi'_- \\ -i\psi'_+ \end{pmatrix} \tag{43}$$

$$\psi'_+\left(x^\mu, \phi, \rho\right) \to \Psi_+\left(x^\mu, \phi, \rho\right) = +i\psi_-\left(x^\mu, \phi, \rho_c - \rho\right) \tag{44}$$

$$\psi'_-\left(x^\mu, \phi, \rho\right) \to \Psi_-\left(x^\mu, \phi, \rho\right) = -i\psi_+\left(x^\mu, \phi, \rho_c - \rho\right) \tag{45}$$

Our periodic boundary condition is

$$\psi_\pm\left(x^\mu, \phi, \rho\right) = \Psi_\pm\left(x^\mu, \rho_c + \rho\right) \tag{46}$$

$$\psi_\pm\left(x^\mu, \phi, \rho\right) = \psi_\pm\left(x^\mu, \phi, 2\rho_c + \rho\right) \tag{47}$$

$$\begin{aligned} \psi_\pm\left(x^\mu, \phi, -\rho\right) &= \Psi_\pm\left(x^\mu, \phi, \rho_c - \rho\right) \\ &= \pm i\psi_\mp\left(x^\mu, \phi, \rho\right) \end{aligned}$$

$$\begin{aligned} \psi_\pm\left(x^\mu, \phi, \rho_c + \rho\right) &= \Psi_\pm\left(x^\mu, \phi, \rho\right) \\ &= \pm i\psi_\mp\left(x^\mu, \phi, \rho_c - \rho\right) \end{aligned}$$

We immediately recognize $\rho = 0, \rho_c$ to be the fixed points of the orbifold.

It is convenient to rewrite $\psi_\pm$ as

$$\psi_\pm = U \pm iV \tag{48}$$

It then follows that

$$\begin{aligned}U\left(x^{\mu},\phi,-\rho\right) &= V\left(x^{\mu},\phi,\rho\right)\\ V\left(x^{\mu},\phi,-\rho\right) &= U\left(x^{\mu},\phi,\rho\right)\end{aligned}$$

and

$$\begin{aligned}U\left(x^{\mu},\phi,\rho+\rho_c\right) &= V\left(x^{\mu},\phi,\rho_c-\rho\right)\\ V\left(x^{\mu},\phi,\rho_c+\rho\right) &= U\left(x^{\mu},\phi,\rho_c-\rho\right)\end{aligned}$$

With

$$S_{\phi}=-\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^{\rho} \tag{49}$$

the $\phi\to-\phi$ symmetry translates to

$$\psi'_{\pm}\left(x^{\mu},\phi,\rho\right)\to\Psi_{\pm}\left(x^{\mu},\phi,\rho\right)=-\gamma^5\psi_{\pm}\left(x^{\mu},\phi_c-\phi,\rho\right) \tag{50}$$

Combine this symmetry with the periodic b.c.

$$\begin{aligned}\psi_{\pm}\left(x^{\mu},\phi,\rho\right) &= \Psi_{\pm}\left(x^{\mu},\phi+\phi_c,\rho\right)\\ &= \psi_{\pm}\left(x^{\mu},2\phi_c+\phi,\rho\right)\end{aligned}$$

and

$$\begin{aligned}\psi_{\pm}\left(x^{\mu},-\phi,\rho\right) &= \Psi_{\pm}\left(x^{\mu},\phi_c-\phi,\rho\right)\\ &= -\gamma_5\psi_{\pm}\left(x^{\mu},\phi_c-\left(\phi_c-\phi\right),\rho\right)\end{aligned}$$

and

$$\begin{aligned}\psi_{\pm}\left(x^{\mu},\phi+\phi_c,\rho\right) &= \Psi_{\pm}\left(x^{\mu},\phi,\rho\right)\\ &= -\gamma_5\psi_{\pm}\left(x^{\mu},\phi_c-\phi,\rho\right)\end{aligned}$$

This shows that $\phi=0,\phi_c$ are fixed points.

One can subsequently define the chiral components of ψ_+,ψ_- by using the usual operators

$$P_{R,L}=\frac{1\pm\gamma^5}{2} \tag{51}$$

with

$$\begin{aligned}\psi_{+,R} &= P_R\psi_+\\ \psi_{+,L} &= P_L\psi_+\\ \psi_{-,R} &= P_R\psi_-\\ \psi_{-,L} &= P_L\psi_-\end{aligned}$$

It follows that

$$
\begin{aligned}
U\left(x^{\mu},-\phi,\rho\right) &= -\gamma_5 U\left(x^{\mu},\phi,\rho\right)\\
V\left(x^{\mu},-\phi,\rho\right) &= -\gamma_5 V\left(x^{\mu},\phi,\rho\right)\\
U\left(x^{\mu},\phi+\phi_c,\rho\right) &= -\gamma_5 U\left(x^{\mu},\phi_c-\phi,\rho\right)\\
V\left(x^{\mu},\phi+\phi_c,\rho\right) &= -\gamma_5 V\left(x^{\mu},\phi-\phi_c,\rho\right)
\end{aligned}
$$

Summarizing all the b.c.:

$$
\begin{aligned}
U_L\left(x^{\mu},-\phi,\rho\right) &= U_L\left(x^{\mu},\phi,\rho\right)\\
U_R\left(x^{\mu},-\phi,\rho\right) &= -U_R\left(x^{\mu},\phi,\rho\right)\\
V_L\left(x^{\mu},-\phi,\rho\right) &= V_L\left(x^{\mu},\phi,\rho\right)\\
V_R\left(x^{\mu},-\phi,\rho\right) &= -V_R\left(x^{\mu},\phi,\rho\right)\\
U_L\left(x^{\mu},\phi+\phi_c,\rho\right) &= U_L\left(x^{\mu},\phi_c-\phi,\rho\right)\\
U_R\left(x^{\mu},\phi+\phi_c,\rho\right) &= -U_R\left(x^{\mu},\phi_c-\phi,\rho\right)\\
V_L\left(x^{\mu},\phi+\phi_c,\rho\right) &= V_L\left(x^{\mu},\phi_c-\phi,\rho\right)\\
V_R\left(x^{\mu},\phi+\phi_c,\rho\right) &= -V_R\left(x^{\mu},\phi_c-\phi,\rho\right)\\
U_L\left(x^{\mu},\phi,-\rho\right) &= V_L\left(x^{\mu},\phi,\rho\right)\\
U_R\left(x^{\mu},\phi,-\rho\right) &= V_R\left(x^{\mu},\phi,\rho\right)\\
V_L\left(x^{\mu},\phi,-\rho\right) &= U_L\left(x^{\mu},\phi,\rho\right)\\
V_R\left(x^{\mu},\phi,-\rho\right) &= U_R\left(x^{\mu},\phi,\rho\right)\\
U_L\left(x^{\mu},\phi,\rho+\rho_c\right) &= V_L\left(x^{\mu},\phi,\rho_c-\rho\right)\\
U_R\left(x^{\mu},\phi,\rho+\rho_c\right) &= V_R\left(x^{\mu},\phi,\rho_c-\rho\right)\\
V_L\left(x^{\mu},\phi,\rho_c+\rho\right) &= U_L\left(x^{\mu},\phi,\rho_c-\rho\right)\\
V_R\left(x^{\mu},\phi,\rho_c+\rho\right) &= U_R\left(x^{\mu},\phi,\rho_c-\rho\right)
\end{aligned}
$$

The reader will note that our representations for the two Z_2 symmetries $\phi \to -\phi$ and $\rho \to -\rho$ acting on the fermions do not commute. Because $[S_\phi, S_\rho] \neq 0$, we cannot find a decomposition of the fermion field ψ such that its components have definite Z_2 transformation properties (even or odd) under both Z_2's. This fact may seem odd, given our intuition that these two Z_2 symmetries are independent. In [**?**], the fermion representation of these Z_2's do commute, and boundary conditions are derived for component fields with definite Z_2 transformation properties under both symmetries. As we demonstrate below, this treatment is fundamentally not possible. It follows that the action is not invariant under both symmetries in the inconsistent representation given in [**?**], as the reader may verify. With incorrect boundary conditions, the resultant solutions are not trustworthy. We now demonstrate why S_ϕ and S_ρ must anticommute.

As is well-known, fermions have the odd property of going into minus themselves when rotated by 2π (pun intended). Applying successively the discrete transformations $\phi \rightarrow -\phi$ followed by $\rho \rightarrow -\rho$ is equivalent to a rotation of π radians in the $\phi - \rho$ plane. Applying these discrete transformations in the opposite order is equivalent to a rotation of $-\pi$ radians in the same plane. The difference in angles of these two rotations is 2π. Applying both transformations, first in the order $S_\rho S_\phi$ and then in the order $S_\phi S_\rho$, is equivalent to a rotation of 2π in the $\phi - \rho$ plane, and must result in an overall minus sign for the fermion field. Therefore we must have $\{S_\phi, S_\rho\} = 0$ in order to be consistent with the spinor nature of fermions.

From these boundary conditions we see that $U_R = V_R = 0$ on the entire boundary of rectangular region of dimensions $\phi_c \times \rho_c$ of our fundamental cell. It also follows that both U_L and V_L are even functions of the ϕ coordinate and that $U_L = V_L$ on the $\rho = 0$ and $\rho = \rho_c$ boundaries.

4 Scalar Field Zero Mode and Quark Mass Matrices

In this section, we solve for the zero mode of a six dimensional Higgs scalar field. As in the fermionic case, we also include a mass term for the scalar in the six dimensional action.

We consider a real scalar field Φ propagating in a six dimensional curved background described by the metric (3). Including a six dimensional mass term, the action is:

$$S = \frac{1}{2} \int \mathrm{d}^4 x \int \mathrm{d}\phi \mathrm{d}\rho \sqrt{G} \left(G^{AB} \partial_A \Phi \partial_B \Phi - m_\Phi^2 \Phi^2 \right) \tag{52}$$

Integrating by parts, one obtains:

$$\begin{aligned} S \quad = \quad & \frac{1}{2} \int \mathrm{d}^4 x \int \mathrm{d}\phi \mathrm{d}\rho \left(\frac{e^\sigma e^\gamma}{\left(e^\sigma + e^\gamma - 1\right)^4} \eta^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \Phi \partial_\phi \left(\frac{e^\gamma}{e^\sigma \left(e^\sigma + e^\gamma - 1\right)^4} \partial_\phi \Phi \right) \right. \\ & \left. + \, \Phi \partial_\rho \left(\frac{e^\sigma}{e^\gamma \left(e^\sigma + e^\gamma - 1\right)^4} \partial_\rho \Phi \right) - m_\Phi^2 \Phi^2 \sqrt{G} \right) \end{aligned}$$

In order to give a four dimensional interpretation to this action, we go through the dimensional reduction procedure. Thus we decompose the six dimensional field into KK modes

$$\Phi\left(x, \phi, \rho\right) = \sum_{n,m} \phi_{n,m}\left(x\right) f_{n,m}\left(\phi, \rho\right) \tag{53}$$

Using this decomposition, the above action becomes a sum over KK modes.

$$S = \frac{1}{2} \sum_{n,m} \int \mathrm{d}^4 x \left\{ \eta^{\mu\nu} \partial_\mu \phi_{n,m}\left(x\right) \partial_\nu \phi_{n,m}\left(x\right) - m_{n,m}^2 \phi_{n,m}^2\left(x\right) \right\} \tag{54}$$

In order to reproduce the canonical four dimensional kinetic terms, we need to impose the orthogonality relations:

$$\int\int \mathrm{d}\phi \mathrm{d}\rho \frac{e^{\sigma}e^{\gamma}}{\left(e^{\sigma}+e^{\gamma}-1\right)^{4}} f^{*}_{m,i}\left(\phi,\rho\right) f_{n,k}\left(\phi,\rho\right) = \delta_{mn}, \delta_{ik} \tag{55}$$

Varying the action with respect to f_n leads to the following equation for the zero mode:

$$\partial_{\phi}\left(\frac{e^{\gamma}}{e^{\sigma}\left(e^{\sigma}+e^{\gamma}-1\right)^{4}}\partial_{\phi}f_0\right)+\partial_{\rho}\left(\frac{e^{\sigma}}{e^{\gamma}\left(e^{\sigma}+e^{\gamma}-1\right)^{4}}\partial_{\rho}f_0\right)-\frac{m^2e^{\sigma}e^{\gamma}}{\left(e^{\sigma}+e^{\gamma}-1\right)^{6}}f_0=0 \tag{56}$$

From the form of this equation, we expect the ϕ and ρ dependence of f_0 to be the same. Hence, we write

$$\begin{aligned}\partial_{\phi}\left(\frac{e^{\gamma}}{e^{\sigma}\left(e^{\sigma}+e^{\gamma}-1\right)^{4}}\partial_{\phi}f_0\right) &= \frac{m^2e^{\sigma}e^{\gamma}}{2\left(e^{\sigma}+e^{\gamma}-1\right)^{6}}f_0\\ \partial_{\rho}\left(\frac{e^{\sigma}}{e^{\gamma}\left(e^{\sigma}+e^{\gamma}-1\right)^{4}}\partial_{\rho}f_0\right) &= \frac{m^2e^{\sigma}e^{\gamma}}{2\left(e^{\sigma}+e^{\gamma}-1\right)^{6}}f_0\end{aligned}$$

Any solution to this pair of equations automatically satisfies the original equation.

The first of the above equations can be written as

$$\left(\frac{4\sigma' e^{\gamma}}{\left(e^{\sigma}+e^{\gamma}-1\right)}+\frac{\sigma' e^{\gamma}}{e^{\sigma}}\right)\partial_{\phi}f_0-\frac{e^{\gamma}}{e^{\sigma}}\partial^2_{\phi}f_0-\frac{m^2e^{\sigma}e^{\gamma}f_0}{2\left(e^{\sigma}+e^{\gamma}-1\right)^{2}}=0 \tag{57}$$

Substituting in the forms for σ and γ, we arrive at the two equations:

$$\left(\frac{4ke^{k\rho}}{\left(e^{k\phi}+e^{k\rho}-1\right)}+\frac{ke^{k\rho}}{e^{k\phi}}\right)\partial_{\phi}f_0-\frac{e^{k\rho}}{e^{k\phi}}\partial^2_{\phi}f_0-\frac{m^2e^{k\phi}e^{k\rho}f_0}{2\left(e^{k\phi}+e^{k\rho}-1\right)^{2}}=0 \tag{58}$$

$$\left(\frac{4ke^{k\sigma}}{\left(e^{k\phi}+e^{k\rho}-1\right)}+\frac{ke^{k\sigma}}{e^{k\rho}}\right)\partial_{\rho}f_0-\frac{e^{k\phi}}{e^{k\rho}}\partial^2_{\rho}f_0-\frac{m^2e^{k\phi}e^{k\rho}f_0}{2\left(e^{k\phi}+e^{k\rho}-1\right)^{2}}=0 \tag{59}$$

For the special case when

$$2k^2m^2 = 25k^4 \tag{60}$$

we can find an analytic solution:

$$f\left(\phi,\rho\right) = e^{\left(\frac{\ln\left(e^{k\phi}+e^{k\rho}-1\right)}{2k^2}\right)} \tag{61}$$

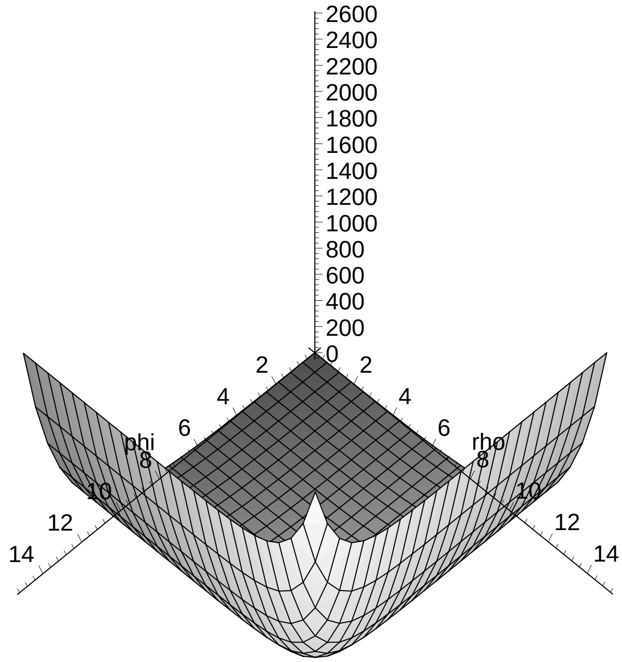

Fig. (4): A plot of the scalar profile for the case $k = 1$.

In Fig. (4) we plot the scalar profile for the case $k = 1$.

Having presented the equations for the fermion field profiles in the extra dimensions and having demonstrated the complex nature of the solutions, we may now see how CP violation emerges naturally. From the fermions and the Higgs scalar, we construct a six dimensional lorentz invariant Yukawa interaction from a term such as

$$\sqrt{-G}\lambda_{ij}^{(6)} H\overline{\psi_i}\psi_j \tag{62}$$

where the fermion fields are eight component objects and their indices i and j are generation labels with the Yukawa coupling matrices $\lambda_{ij}^{(6)}$ connecting only fermions in combinations consistent with gauge group invariance. The fermion bilinear may be expressed as

$$\begin{aligned}\overline{\psi}\psi &= -i\overline{\psi_-}\psi_+ + i\overline{\psi_+}\psi_- \\ &= i\overline{\psi_{L+}}\psi_{R-} + i\overline{\psi_{R+}}\psi_{L-} \\ &\quad -i\overline{\psi_{L-}}\psi_{R+} - i\overline{\psi_{R-}}\psi_{L+}\end{aligned}$$

where the conjugate fields are formed with the usual γ_0 of the four dimensional Dirac algebra.

One may write the exact same form of the Yukawa coupling in the six dimensional action as in the SM action:

$$\int \mathrm{d}^4x \int \mathrm{d}\phi \mathrm{d}\rho \sqrt{-G}\lambda_{ij}^{(6)} H\overline{\psi_i}\psi_j \tag{63}$$

Recall, however, that the motivation for considering the implications of extra dimensions is to see if their existence can help to reduce or simplify the redundancy inherent in the four dimensional Yukawa coupling matrices. Allowing $\lambda_{ij}^{(6)}$ and the six dimensional fermion mass terms m_i to be arbitrary parameters only increases the redundancy of physical information contained in the parameters of the action, and, from the perspective of attempting to gain deeper understanding of the fermion mass hierarchy, considerably weakens the motivation of considering extra dimensions. However, one starting point often adopted in attempts to study the fermion mass hierarchy is to start from a discrete flavor symmetry which leads to the so-called democratic mass matrix, with all entries being the same. This simplest ansätz leads to one massive eigenstate and two degenerate massless eigenstates and is thus considered a reasonably successful approximation given the simplicity of the ansätz. In [**?**], the introduction of a single flat extra dimension provides a theoretical explanation of the democratic ansätz itself in terms of higher dimensional geography rather than some additional flavor physics. The introduction of an additional flat extra dimension and its associated Dirac algebra then allow for the possibility to realize a more realistic spectrum than that provided by the democratic mass matrices. Because the extra dimensions are taken to be flat, the fermion profiles may be separable functions of each of the extra dimensions. This property transforms the effective democratic quark mass matrices one obtains from dimensionally reducing from five to four dimensions into pure phase mass matrices.

The success of this approach in the flat space scenario cannot be carried over without modification to the case of two warped extra dimensions. Because the equations for the fermion profiles in the warped scenario we are considering are not separable, it is not possible to automatically achieve pure phase effective four dimensional quark mass matrices within our context.

If we set $\lambda_{ij}^{(6)}$ equal to the democratic mass matrix in both the up and down quark sectors, we arrive at essentially the same level of understanding as when the democratic form is adopted in flavor symmetry approaches to the problem in four dimensions. In that case, a longstanding difficulty has been to find a successful implementation of a breaking of this symmetry, (sometimes involving the additional physics of a "flavon" field) with the additional complication of ultimately arriving at complex mass matrices.

The principle observation of this work is that the presence of the extra dimensions serves to break the democratic form of the $\lambda_{ij}^{(6)}$, which is now controlled by the six dimensional fermion mass terms down to the effective $\lambda_{ij}^{(4)}$. Geography in the extra warped dimensions is an alternative to the flavon field method of breaking the flavor symmetry of the Yukawa terms. In addition, this geometric method of breaking the flavor symmetry naturally leads to complex effective four dimensional mass matrices.

Adopting the democratic form for $\lambda_{ij}^{(6)}$ in each quark sector does not result in a calculable model of flavor mixing and masses. It does, however, correspond to a so-called minimal parameter basis. Six quark masses and four flavor mixing parameters are derived from ten six-dimensional parameters. This minimal parameter set arises from the following considerations. The SM gauge symmetries allow for the existence of nine different six-dimensional fermion masses m_i. The left-handed up-type and down-type quarks of the same generation must have the same six-dimensional mass parameter m because together they form an $SU(2)$ doublet. The right-handed components of the up and down type quarks for each generation have different mass parameters. From the four dimensional perspective, a massive fermion field is formed only after electroweak symmetry breaking. In addition to these nine independent mass parameters, we also have the freedom associated with the curvature k of the bulk AdS space. We are assuming equal magnitudes for all the 4-brane tensions and so have only one dimension-full parameter k, instead of two as would be the case if we allowed arbitrary 4-brane tensions. Requiring that this setup reproduce the RS resolution of the gauge hierarchy problem then fixes the coordinate length of the fundamental cell we are considering.

5 Conclusion

We first remind the reader of some of the promising results already attained in addressing the fermion mass and mixing hierarchy problems within the context of extra dimensions [**?**, **?**, **?**, **?**]. Considering the 2×2 matrix with columns labelled "one extra dimension" and "two extra dimensions" and rows labelled "flat extra space" and "warped extra space", we have shown that the $(2,2)$ element is also a possible arena in which to study the problem. But the principle motivations for considering the two warped extra dimensions scenario are not just for the sake of completeness. The possibility of attaining the CP violating phase in the quark flavor-mixing matrix via this higher dimensional mechanism is a property of the Dirac algebra in six dimensions and remains a possibility whether or not the two extra dimensions are flat or warped. Independent of considerations of the gauge hierarchy problem, one advantage of the warped case over the flat case is that no additional physics is required in order to localize the fermions. In the flat space case, the localization mechanism of the fermions involves the introduction of a scalar field and specific forms for the scalar field must be assumed. Even then the fermion profiles are solved analytically only after making approximations to the given scalar field [**?**].

We have shown in this work that CP violation may be understood as a natural consequence of the Dirac algebra in six dimensions. Another motivation for going to six dimensions concerns the mystery of the generation index. As shown in [**?**], multibrane world scenarios imply the existence of light KK modes that are suggestive of family replication. Going to six dimensions may alleviate some of the difficulties encountered in trying to implement this program. This possibility is currently under investigation.

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